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Eigenvalue density in Hermitian matrix models by the Lax pair method

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Abstract

In this paper, a new method is discussed to derive the eigenvalue density in a Hermitian matrix model with a general potential. The density is considered on one interval or multiple disjoint intervals. The method is based on Lax pair theory and the Cayley–Hamilton theorem by studying the orthogonal polynomials associated with the Hermitian matrix model. It is obtained that the restriction conditions for the parameters in the density are connected to the discrete Painlevé I equation, and the results are related to the scalar Riemann–Hilbert problem. Some special density functions are also discussed in association with the known results in this subject.

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1. Introduction

This paper is a continuation of the previous works [1–4] about the linearized equation $d^2\phi/d\eta^2 = -\xi^2 F(\eta,\xi)\phi$ for the Painlevé or discrete Painlevé equations. The connection between the integral $\int_{\eta_0}^{\eta}\sqrt{F(t,\xi)}\,dt$ in the WKB asymptotics and the analytic potential in the previous researches is now extended to the relation between $\sqrt{F(\eta,\xi)}$ and the derivative of the potential function in the complex plane to investigate the distribution of eigenvalues considered in matrix models. The fundamental density in this consideration is the Wigner semicircle obtained from a differential equation for the Hermite polynomials, similar to the linearized equation above, as discussed in [5]. The differential equation and the recursion formula for the Hermite polynomials form a degenerate case of the Lax pair for the discrete Painlevé I equation, and then the Lax pair theory is now applied to study a general density problem.

The eigenvalue density is the solution of the energy minimization problem for a given potential of the model, and there have been various methods developed in history and specially in recent years to solve this type of problems, such as the Plemelj formula or Riemann–Hilbert problem related methods. This report is to show that a new algebraic method can be developed to calculate the densities by using the Lax pair theory and discrete Painlevé equations. The factorization of the reduced matrix from the Lax pair by applying the Cayley–Hamilton theorem can simplify the analytic calculations when working on the density and the consequent problems as explained in the following.

Consider the Hermitian matrix model with a general potential $V(z) = \sum_{j=0}^{2m} t_j z^j$, where z is a real or complex variable, t_j are real, and $t_{2m} > 0$ to have the convergent integral for the partition function

$$Z_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{e}^{-\Sigma_{i=1}^n V(z_i)} \prod_{j<k} (z_j - z_k)^2 \, \mathrm{d}z_1 \cdots \mathrm{d}z_n.$$

The free energy function is defined as [6] $E^{(0)} = -\lim_{n\to\infty} \frac{1}{n^2} \ln Z_n$. By the scaling transformation $z = n^{\frac{1}{2m}} \eta$ and $t_j = n^{1-\frac{j}{2m}} g_j$, the potential becomes $W(\eta) = \sum_{j=0}^{2m} g_j \eta^j$. The eigenvalue density $\rho_m(\eta)$ on ν_1 interval(s) $\Omega = \cup_{j=1}^{\nu_1} \big[\eta_-^{(j)}, \eta_+^{(j)}\big]$ is defined to minimize the free energy function

$$E^{(0)} = \int_\Omega W(\eta)\rho_m(\eta)\,\mathrm{d}\eta - \int_\Omega \int_\Omega \ln|\lambda - \eta| \rho_m(\lambda)\rho_m(\eta)\,\mathrm{d}\lambda\,\mathrm{d}\eta. \tag{1.1}$$

The density is required to satisfy the following conditions [5, 6]:

(i) ρ_m is non-negative when $\eta \in \Omega$,

$$\rho_m(\eta) \geqslant 0; \tag{1.2}$$

(ii) ρ_m is normalized,

$$\int_\Omega \rho_m(\eta)\,\mathrm{d}\eta = 1; \tag{1.3}$$

(iii) ρ_m satisfies a variational equation when η is an inner point of Ω,

$$(\mathrm{P}) \int_\Omega \frac{\rho_m(\lambda)}{\eta - \lambda}\,\mathrm{d}\lambda = \frac{1}{2} W'(\eta), \tag{1.4}$$

where (P) stands for the principal value of the integral. So the problem is to find $\rho_m(\eta)$ such that it satisfies these three conditions. The density generally takes a form as the product of a polynomial and the square root of another polynomial as introduced in the following, and the nonlinear relation(s) satisfied by the parameters in the density will become complicated as the order of the potential and the number of the potential parameters are increasing.

When $m = 1$ and $W(\eta) = \eta^2$, there is $\rho_1(\eta) = \frac{1}{\pi}\sqrt{2 - \eta^2}$, for $\eta \in [-\sqrt{2}, \sqrt{2}]$, which is the well-known Wigner semicircle. When $m = 2$ and $W(\eta) = \frac{1}{2}\eta^2 + g\eta^4$, it is given in [6] that

$$\rho_2(\eta) = \frac{1}{\pi}\left(\frac{1}{2} + 4gb^2 + 2g\eta^2\right)\sqrt{4b^2 - \eta^2}, \tag{1.5}$$

for $\eta \in [-2b, 2b]$, where

$$b^2 + 12gb^4 = 1. \tag{1.6}$$

The free energy function is

$$E^{(0)}(g) = E^{(0)}(0) + \tfrac{1}{24}(b^2 - 1)(9 - b^2) - \tfrac{1}{2}\ln b^2. \tag{1.7}$$

It can be calculated that $E^{(0)}(0) = 3/4$. And $E^{(0)}$ has a singular point at $g = g_c$, where $g_c = -1/48$. See [6–9] for the details.

When $W(\eta) = g_{2m}\eta^{2m}$, there is [10, 11]

$$\rho_m(\eta) = \frac{1}{\pi} m g_{2m} h(\eta)\sqrt{4b^2 - \eta^2}, \tag{1.8}$$

for $\eta \in [-2b, 2b]$, where

$$h(\eta) = \eta^{2m-2} + \sum_{p=1}^{m-1} \eta^{2m-2-2p}(2b)^{2p} \prod_{l=1}^{p} \frac{2l-1}{2l}, \tag{1.9}$$

subject to the condition

$$mg_{2m}(2b)^{2m} \prod_{l=1}^{m} \frac{2l-1}{2l} = 1. \tag{1.10}$$

More results can be found, for instance, in [10–18]. Being part of their works, the density and free energy for the potential $\frac{1}{2}\eta^2 + g_{2m}\eta^{2m}$ studied in [14] using the combinatoric method are a generalization of the results discussed above. In [15], a density function of the form

$$\rho_{m+1}(\eta) = c_0^{-1}(\eta - c)^{2m}\sqrt{4-\eta^2}, \qquad c_0 = \int_{-\infty}^{\infty} (\eta - c)^{2m}\sqrt{4-\eta^2}\,\mathrm{d}\eta, \tag{1.11}$$

is given and applied to study a third-order phase transition problem by extending the density from one interval to multiple disjoint intervals. The critical point for the transition in [15] is chosen as the discrete system is changed to the continuum Painlevé II system. The string equations (2.9) and (2.10) in [15] for the coefficients β_n and γ_n in the recursion formula $\eta\psi_n = \gamma_{n+1}\psi_{n+1} + \beta_n\psi_n + \gamma_n\psi_{n-1}$ in [15] are related to the discrete Painlevé I equation in this paper.

The density and the conditions for the parameters in this paper are obtained from the Lax pair structure and the discrete Painlevé I equation as outlined in the following. Consider the orthogonal polynomials $p_n = z^n + \cdots$ on the real line with the weight $\exp(-V(z))$: $\langle p_n, p_{n'}\rangle = h_n\delta_{n,n'}$. By using the recursion formula [19] $p_{n+1}(z) + u_n p_n(z) + v_n p_{n-1}(z) = zp_n(z)$, it will be discussed that $\Phi_n(z) = \mathrm{e}^{-\frac{1}{2}V(z)}(p_n(z), p_{n-1}(z))^T$ satisfies two equations, $\Phi_{n+1} = L_n\Phi_n$, and $\frac{\partial}{\partial z}\Phi_n = A_n\Phi_n$. These two equations are called the Lax pair for the discrete Painlevé I equation which is a set of two discrete equations for u_n and v_n: $\langle p_n, V'p_{n-1}\rangle = nh_{n-1}$, and $\langle p_n, V'p_n\rangle = 0$, where $h_n/h_{n-1} = v_n$. These two relations will be applied to derive the conditions for the parameters in the density.

The coefficient matrix $A_n(z)$ in the above equation is generally a complicated 2×2 matrix. Replacing u_{n-l} and v_{n-l+1} in A_n by new parameters x_n and y_n respectively for a range of l, we can get another matrix

$$\tilde{A}_n(z) = D_n\tilde{F}_n(z)D_n^{-1} - \tfrac{1}{2}V'(z)I, \tag{1.12}$$

where the matrix $\tilde{F}_n(z)$ is a polynomial of the matrix J_n derived from L_n,

$$J_n = \begin{pmatrix} 0 & 1 \\ -y_n & z - x_n \end{pmatrix}.$$

Here $D_n = \mathrm{diag}(h_n, h_{n-1})$, and I is the identity matrix. By the Cayley–Hamilton theorem for J_n, there is $(z - x_n)I = J_n + y_nJ_n^{-1}$. Applying this relation to $V'(z)I$ in (1.12), the matrix $D_n^{-1}\tilde{A}_n(z)D_n$ can be factorized as a product of a polynomial and a simple matrix

$$D_n^{-1}\tilde{A}_n(z)D_n = f_{2m-2}(z)\big(J_n(z) - y_nJ_n^{-1}(z)\big), \tag{1.13}$$

where the polynomial $f_{2m-2}(z)$ will be given in section 3. There is an important asymptotics

$$\sqrt{-\det\tilde{A}_n(z)} = \frac{1}{2}V'(z) - \frac{n}{z} + O\left(\frac{1}{z^2}\right), \tag{1.14}$$

as $z \to \infty$ in the complex plane, derived by referring the structure of the discrete Painlevé I equation. This property will be finally used to satisfy the conditions (1.3) and (1.4). If

$z/n^{\frac{1}{2m}}$, $t_j/n^{1-\frac{j}{2m}}$, $x_n/n^{\frac{1}{2m}}$ and $y_n/n^{\frac{1}{m}}$ are denoted as η, g_j, a and b^2 respectively, the formula for $\rho_m(\eta)$ on the interval $[\eta_-, \eta_+] = [a-2b, a+2b]$ can be obtained by

$$\frac{1}{n\pi}\sqrt{\det \tilde{A}_n(z)}\,\mathrm{d}z = \rho_m(\eta)\,\mathrm{d}\eta. \tag{1.15}$$

The eigenvalue density problem is then solved when condition (1.2) is satisfied.

The density results can be applied to get the free energy which is an important physical quantity to study the nonlinear properties as considered in the expansion theory. The free energy and consequent physical quantities, such as internal energy and specific heat, are generally studied based on the logarithmic partition function by using Wilson loops and topological methods in physics. Researches in this field include, for instance, planar diagrams [6–9], phase transitions [15, 20, 21], graphical enumeration [13, 14], and continuum limit and combinatoric interpretations [14–18]. The Lax pair method here provides another technique to handle the branch singularities when computing the free energy function as shown in section 6 for the one interval case.

Other models can also be studied by the Lax pair method as seen in the appendix. The weak- and strong-coupling densities in the unitary matrix model [20] can be derived using the Lax pair for the discrete Painlevé II equation [4] associated with the orthogonal polynomials on unit circle. The density in [22, 23] can be obtained using the Laguerre polynomials. It would be interesting to investigate in the future whether more results for the Lax pair and discrete Painlevé equations associated with the orthogonal polynomials obtained in the literatures, such as [24–34] and the references therein, can be applied to study the density problems in matrix models or random matrix ensembles.

This paper is organized as follows. To avoid the symbolic complexity, we just show the details for the density on one interval, and point out some key steps for the multiple interval case in sections 2.2 and 5, plus an example in section 7.1. In the next section, we will start from the orthogonal polynomials associated with the Hermitian matrix model to derive the Lax pair and discrete Painlevé I equation, and the matrix $\tilde{A}_n$ is then defined. In section 3, $\tilde{A}_n$ is factorized by using the Cayley–Hamilton theorem. The factorization property will give the formula for the density by scaling. In section 4, the asymptotics for $(-\det \tilde{A}_n)^{1/2}$ and $(-\det A_n)^{1/2}$ as $z \to \infty$ in the complex plane are obtained. In section 5, we will discuss the density and the related scalar Riemann–Hilbert problem. In section 6, the general free energy function for one interval case is discussed. In section 7, some special densities are presented based on the general results, including some symmetric densities associated with the results in other literatures. The appendix is about some density functions in econophysics and the unitary matrix model.

2. Lax pair and discrete Painlevé I equation

2.1. Lax pair and the orthogonal polynomials

It is discussed in the introduction that $\rho_m(\eta)$ on one interval $[\eta_-, \eta_+]$, for instance, needs to satisfy the conditions $\int_{\eta_-}^{\eta_+} \rho_m(\eta)\,\mathrm{d}\eta = 1$, and

$$\frac{1}{2}W'(\eta) = (\mathrm{P})\int_{\eta_-}^{\eta_+} \frac{\rho_m(\lambda)}{\eta-\lambda}\,\mathrm{d}\lambda \doteq \lim_{\epsilon\to 0}\left(\int_{\eta_-}^{\eta-\epsilon} \frac{\rho_m(\lambda)}{\eta-\lambda}\,\mathrm{d}\lambda + \int_{\eta+\epsilon}^{\eta_+} \frac{\rho_m(\lambda)}{\eta-\lambda}\,\mathrm{d}\lambda\right),$$

for $\eta \in (\eta_-, \eta_+)$. The method is to search an analytic function with asymptotics $\frac{1}{2}W'(\eta) - \frac{1}{\eta}$, as $\eta \to \infty$ in the complex plane. Then by the contour integral method, these two conditions can be satisfied.

To have such asymptotics, consider the orthogonal polynomials $p_n(z) = z^n + \cdots$ on $(-\infty, \infty)$ associated with the Hermitian matrix model, defined by

$$\langle p_n, p_{n'}\rangle \equiv \int_{-\infty}^{\infty} p_n(z)p_{n'}(z)\,\mathrm{e}^{-V(z)}\,\mathrm{d}z = h_n\delta_{n,n'}, \tag{2.1}$$

where $V(z) = \sum_{j=0}^{2m} t_j z^j$, $t_{2m} > 0$. The basic asymptotics $\mathrm{e}^{-V(z)/2}p_n(z) \sim \mathrm{e}^{-\frac{1}{2}V(z)+n\ln z}$(as $z \to \infty$) leads an idea to use the differential equation of the polynomials to derive the density formula. In the following, the Lax pair is introduced in terms of the orthogonal polynomials given above.

The orthogonal polynomials satisfy a recursion formula [19]:

$$p_{n+1}(z) + u_n p_n(z) + v_n p_{n-1}(z) = zp_n(z). \tag{2.2}$$

By multiplying $p_{n-1}(z)\,\mathrm{e}^{-V(z)}$ on both sides of this recursion formula and taking integral, we have $v_n = h_n/h_{n-1}$. This recursion formula will give the first equation of the Lax pair. For the second equation of the pair, let us consider the differential equation.

When $n \geqslant 2m-1$, express the derivative of p_n with respect to z as a linear combination of p_j, $j = 0, 1, \ldots, n-1$,

$$\frac{\partial}{\partial z}p_n = a_{n,n-1}p_{n-1} + a_{n,n-2}p_{n-2} + \cdots + a_{n,0}p_0, \tag{2.3}$$

where $a_{n,j}$ are independent of z. By integration by parts, there are

$$a_{n,j}h_j = \int_{-\infty}^{\infty} V'(z)p_j(z)p_n(z)\,\mathrm{e}^{-V(z)}\,\mathrm{d}z \qquad ('= \partial/\partial z),$$

for $j = 0, 1, \ldots, n-1$, and $a_{n,j} = 0$ when $j < n-2m+1$ by the orthogonality. Then, by the recursion formula, $\frac{\partial}{\partial z}p_n$ can become as a linear combination of p_n and p_{n-1}, but the new coefficients are dependent on z.

Denote $\Phi_n(z) = \mathrm{e}^{-\frac{1}{2}V(z)}(p_n(z), p_{n-1}(z))^T$. By the above discussions, there are

$$\Phi_{n+1} = L_n\Phi_n, \tag{2.4}$$

where

$$L_n = \begin{pmatrix} z-u_n & -v_n \\ 1 & 0 \end{pmatrix},$$

and

$$\frac{\partial}{\partial z}\Phi_n = A_n(z)\Phi_n, \tag{2.5}$$

for a matrix $A_n(z)$. Equations (2.4) and (2.5) are called the Lax pair for the discrete Painlevé I equation to be discussed in section 2.3, and the structure was given in [26], as well as in [25] (Part 2, chapter 1).

The method in this paper starts from the construction of the matrix A_n. For $m \geqslant 1$ and $n \geqslant 2m$, consider

$$\frac{\partial}{\partial z}p_n = a_{n,n-1}p_{n-1} + a_{n,n-2}p_{n-2} + \cdots + a_{n,n-2m+1}p_{n-2m+1},$$
$$\frac{\partial}{\partial z}p_{n-1} = a_{n-1,n-2}p_{n-2} + a_{n-1,n-3}p_{n-3} + \cdots + a_{n-1,n-2m}p_{n-2m},$$

where, for $n' = n$ or $n-1$, and $k = 1, 2, \ldots, 2m-1$,

$$a_{n',n'-k}h_{n'-k} = \int_{-\infty}^{\infty} V'(z)p_{n'-k}p_{n'}\,\mathrm{e}^{-V(z)}\,\mathrm{d}z. \tag{2.6}$$

It follows that

$$\frac{\partial}{\partial z}\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \sum_{k=1}^{2m-1} C_{n-k}\begin{pmatrix} P_{n-k} \\ P_{n-k-1} \end{pmatrix},$$

where

$$C_{n-k} = \begin{pmatrix} a_{n,n-k}h_{n-k} & 0 \\ 0 & a_{n-1,n-k+1}h_{n-k+1} \end{pmatrix},$$

for $k = 1, \ldots, 2m-1$. And $P_j = p_j/h_j$ satisfy

$$\begin{pmatrix} P_j \\ P_{j-1} \end{pmatrix} = \bar{J}_{j+1}\begin{pmatrix} P_{j+1} \\ P_j \end{pmatrix}, \qquad \bar{J}_{j+1} = \begin{pmatrix} 0 & 1 \\ -v_{j+1} & z - u_j \end{pmatrix},$$

by using (2.4) and $v_{j+1} = h_{j+1}/h_j$. Let $D_n = \mathrm{diag}(h_n, h_{n-1})$. The above discussion gives

$$\frac{\partial}{\partial z}\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = D_n F_n D_n^{-1}\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix},$$

where the matrix F_n is defined by

$$D_n F_n = C_{n-1}\bar{J}_n + \cdots + C_{n-2m+1}\bar{J}_{n-2m+2}\bar{J}_{n-2m+3}\cdots\bar{J}_n. \tag{2.7}$$

Let I be the 2×2 identity matrix. Then, there is

$$A_n = D_n F_n D_n^{-1} - \tfrac{1}{2}V'(z)I, \qquad n \geqslant 2m. \tag{2.8}$$

2.2. Reduced matrices from the Lax pair structure

Let Δ be the operator for index change acting only on the polynomials $\Delta^l p_n = p_{n+l}$, where l is integer. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} p_{n'}\sum_{j=1}^{2m} jt_j(x_n + \Delta + y_n\Delta^{-1})^{j-1}p_{n'-k}\,\mathrm{e}^{-V(z)}\,\mathrm{d}z \\
&\quad = \int_{-\infty}^{\infty} p_{n'}\sum_{j=1}^{2m} jt_j\sum_{q=0}^{j-1}\binom{j-1}{q}x_n^{j-q-1}(\Delta + y_n\Delta^{-1})^q p_{n'-k}\,\mathrm{e}^{-V(z)}\,\mathrm{d}z \\
&\quad = \sum_{j=1}^{2m} jt_j\sum_{q=0}^{j-1}\binom{j-1}{q}x_n^{j-q-1}\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r}y_n^r h_{n'}\delta_{q-k-2r,0},
\end{aligned} \tag{2.9}$$

for $n' = n$ or $n-1, k = 1, 2, \ldots, 2m-1$, and $q = 0, 1, \ldots, 2m-1$, where the new parameters x_n and y_n are introduced by referring the roles of the u_n and v_n in the Lax pair. Here $\binom{q}{r} = q!/(r!(q-r)!)$, $[\cdot]$ denotes the integer part, $\mu_q = (1+(-1)^q)/2$ and $q = 2[q/2]-\mu_q+1$. For $k > 0$, there is $q-k-2r = 2([q/2]-\mu_q-r)+1+\mu_q-k < 0$ if $[q/2]-\mu_q < r$, which implies $\delta_{q-k-2r,0} = 0$ when $r > [q/2]-\mu_q$.

Let

$$\tilde{F}_n = \sum_{j=1}^{2m} jt_j\sum_{q=1}^{j-1}\binom{j-1}{q}x_n^{j-q-1}\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r}y_n^r J_n^{q-2r}, \tag{2.10}$$

where

$$J_n = \begin{pmatrix} 0 & 1 \\ -y_n & z - x_n \end{pmatrix}. \tag{2.11}$$

Define

$$\tilde{A}_n(z) = D_n \tilde{F}_n D_n^{-1} - \tfrac{1}{2} V'(z) I, \tag{2.12}$$

which is a matrix reduced from the Lax pair structure, to be used for the one interval problem.

For the density on disjoint intervals, let

$$J_n^{(\nu)} = \begin{pmatrix} 0 & 1 \\ -y_n^{(1)} & z - x_n^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -y_n^{(\nu)} & z - x_n^{(\nu)} \end{pmatrix}. \tag{2.13}$$

According to the Cayley–Hamilton theorem for $J^{(\nu)}$, there is

$$\left(\mathrm{tr} J_n^{(\nu)}\right) I = J_n^{(\nu)} + \left(\det J_n^{(\nu)}\right) J_n^{(\nu)-1}. \tag{2.14}$$

We can transform t_j $(j = 1, \ldots, 2m)$ into a new set of parameters t_j' $(j = 1, \ldots, 2m)$ by a linear transformation, such that

$$V'(z) = \sum_{s=0}^{\nu-1} z^s \sum_{q=0}^{m_s} t'_{\nu q+s} \left(\mathrm{tr}\, J_n^{(\nu)}\right)^q, \tag{2.15}$$

for some integers m_s $(s = 0, \ldots, \nu - 1)$, where each m_s is the largest integer such that $s + \nu m_s \leqslant 2m - 1$. In fact, by expanding the above expression in terms of z and comparing the coefficients with $V'(z) = \sum_{j=1}^{2m} j t_j z^{j-1}$, we can get a upper triangle matrix T_{2m-1} so that $T_{2m-1}\vec{t'} = \vec{t}$ with $\vec{t} = (t_1, 2t_2, \ldots, 2mt_{2m-1})^T$ and $\vec{t'} = (t_1', \ldots, t_{2m-1}')^T$. The derivative $\partial p_n/\partial z$ is now expanded as

$$\frac{\partial p_n}{\partial z} = \sum_{s=0}^{\nu-1} \sum_{q'=1}^{N_0} a_{n,n-\nu q'+s}^{(\nu)} z^s p_{n-\nu q'}(z) + \sum_{k=\nu N_0+1}^{n} a_{n,n-k}^{(\nu)} p_{n-k}(z), \tag{2.16}$$

where $n - \nu N_0 < \nu$ and the choice of N_0 is dependent on the value of m.

By the index change operator Δ, there is

$$\begin{aligned}\sum_{q=1}^{m_s} t'_{\nu q+s} \int_{-\infty}^{\infty} & p_{n-\nu q'+s} z^s (\Delta^\nu + (\det J_n)\Delta^{-\nu})^q p_n \, \mathrm{e}^{-V(z)} \, \mathrm{d}z \\ &= \sum_{q=1}^{m_s} t'_{\nu q+s} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} \left(\det J_n^{(\nu)}\right)^{q-r} \delta_{q-q'-2r,0}, \qquad q' \leqslant m_s. \end{aligned} \tag{2.17}$$

Then we get another reduced matrix

$$\tilde{A}_n^{(\nu)}(z) = D_n \tilde{F}_n^{(\nu)} D_n^{-1} - \tfrac{1}{2} V'(z) I, \tag{2.18}$$

where

$$\tilde{F}_n^{(\nu)} = \sum_{s=0}^{\nu-1} z^s \sum_{q=1}^{m_s} t'_{\nu q+s} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} \left(\det J_n^{(\nu)}\right)^r \left(J_n^{(\nu)}\right)^{q-2r}, \tag{2.19}$$

by referring that $(p_{n-\nu q'}, p_{n-\nu q'-1})^T$ is connected to

$$D_n \left(\det J_n^{(\nu)}\right)^{-q'} \left(J_n^{(\nu)}\right)^{q'} D_n^{-1} (p_n, p_{n-1})^T.$$

The formula for the matrix $\tilde{A}_n^{(\nu)}(z)$ will be applied to study the density on multiple disjoint intervals as discussed in section 5.

2.3. Reduced equations from discrete Painlevé I

The discrete Painlevé I equation associated with the orthogonal polynomials in the considerations was introduced in [26] as an equation for v_n. As an extension, the discrete Painlevé I equation here is a set of two equations for u_n and v_n.

By orthogonality of the polynomials $p_n(z) = z^n + \cdots$ and integration by parts, there are

$$\langle p_n(z), V'(z)p_{n-1}(z)\rangle = nh_{n-1}, \tag{2.20}$$

$$\langle p_n(z), V'(z)p_n(z)\rangle = 0. \tag{2.21}$$

These two equations are recursion formulas for the parameters u_n and v_n. The set of (2.20) and (2.21) is called the discrete Painlevé I equation when $m = 2$, and called the high-order discrete Painlevé I equation when $m > 2$. The discrete Painlevé I equation is the consistency condition for the Lax pair (2.4) and (2.5). The consistency can be discussed by the methods in the references cited in the introduction. In this paper, only the equations are needed for restricting the parameters.

If the differential equation is written in the form

$$\frac{\partial}{\partial z}p_n = a_{n,n}p_n + a_{n,n-1}p_{n-1} + \cdots + a_{n,n-2m+1}p_{n-2m+1},$$

where $a_{n,n} = 0$, then equation (2.6) is still true for $k = 0$. Write (2.20) and (2.21) as

$$a_{n,n-1}h_{n-1} = nh_{n-1}, \tag{2.22}$$

$$a_{n,n}h_n = 0. \tag{2.23}$$

Based on (2.9) for $n' = n$, $k = 1$ and $k = 0$ respectively, in this method for eigenvalue density on one interval, restrict x_n and y_n to satisfy the following equations:

$$\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^{r+1}\delta_{q,2r+1} = n, \tag{2.24}$$

$$\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^{r}\delta_{q,2r} = 0. \tag{2.25}$$

Note that $\delta_{q,2r+1} = 0$ when q is even, and $\delta_{q,2r} = 0$ when q is odd. After substitutions $q = 2p+1$, $r = p$ in (2.24), and $q = 2p$, $r = p$ in (2.25), there are

$$\sum_{j=2}^{2m} jt_j \sum_{p=0}^{[\frac{j}{2}]-1} \binom{j-1}{2p+1}\binom{2p+1}{p} x_n^{j-2p-2}y_n^{p+1} = n, \tag{2.26}$$

$$\sum_{j=1}^{2m} jt_j \sum_{p=0}^{[\frac{j-1}{2}]} \binom{j-1}{2p}\binom{2p}{p} x_n^{j-2p-1}y_n^{p} = 0. \tag{2.27}$$

These two equations will be changed to get the restriction conditions for the parameters in the density.

Specially, when $V(z)$ is even, $V(-z) = V(z)$, or $t_1 = t_3 = \cdots = t_{2m-1} = 0$, there is $p_n(-z) = p_n(z)$, which implies that $u_n = 0$, and it follows that $x_n = 0$. Then (2.27) becomes $0 = 0$, and (2.26) becomes

$$\sum_{j=1}^{m} 2jt_{2j} \binom{2j-1}{j} y_n^j = n, \tag{2.28}$$

by replacing j by $2j$, and taking $p = j - 1$ on the left-hand side of (2.26). The relations between the parameters are fundamental when studying the nonlinear properties of the density problem as explained before, and relevant discussions can be seen in [14, 17] (section 5.11), for instance. In [14], an enumeration method is applied to derive a parameter relation formula similar to equation (2.28).

3. Factorization of $\tilde{A}_n(z)$

Lemma 3.1. *If x_n, y_n, and t_j $(j = 1, \ldots, 2m)$ satisfy equation (2.27), then for $\tilde{A}_n(z)$ defined by (2.12) and $\mu_q = (1 + (-1)^q)/2$, there is*

$$D_n^{-1}\tilde{A}_n D_n = \frac{1}{2}\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r} y_n^r \left(J_n^{q-2r} - \left(y_n J_n^{-1}\right)^{q-2r}\right). \tag{3.1}$$

Proof. Because

$$(z - x_n)I = J_n + y_n J_n^{-1}, \tag{3.2}$$

and $q = 2[q/2] - \mu_q + 1$, the binomial expansion implies

$$\begin{aligned}
(z - x_n)^q I &= \left(\sum_{r=0}^{[q/2]-\mu_q} + \mu_q \sum_{r=[q/2]}^{[q/2]} + \sum_{r=[q/2]+1}^{2[q/2]-\mu_q+1}\right)\binom{q}{r} y_n^r J_n^{q-2r} \\
&= \sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r} y_n^r J_n^{q-2r} + \mu_q \binom{q}{[q/2]} y_n^{[q/2]} J_n^{q-2[q/2]} \\
&\quad + \sum_{s=0}^{[q/2]-\mu_q}\binom{q}{s} y_n^{q-s} J_n^{-q+2s} \\
&= \sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r} y_n^r \left(J_n^{q-2r} + \left(y_n J_n^{-1}\right)^{q-2r}\right) + \mu_q \binom{q}{[q/2]} y_n^{[q/2]} J_n^{q-2[q/2]},
\end{aligned}$$

where s comes out by substitution $r = q - s$, and is replaced by r in the last step. Since

$$V'(z) = \sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1}(z - x_n)^q,$$

$V'(z)I$ now can be expressed in terms of J_n.

By $D_n^{-1}\tilde{A}_n D_n = \tilde{F}_n - \frac{1}{2}V'(z)I$, where $\tilde{F}_n$ is given by (2.10), we then have

$$\begin{aligned}
D_n^{-1}\tilde{A}_n D_n &= \frac{1}{2}\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r} y_n^r \left(J_n^{q-2r} - \left(y_n J_n^{-1}\right)^{q-2r}\right) \\
&\quad - \frac{1}{2}\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1} \mu_q \binom{q}{[q/2]} y_n^{[q/2]} J_n^{q-2[q/2]}.
\end{aligned}$$

Since $\mu_q = 1$ when q is even, and $\mu_q = 0$ when q is odd, the last part in the above vanishes by taking $q = 2p$ and applying equation (2.27). So the lemma is proved. □

Let

$$\alpha_n = \frac{z - x_n + \sqrt{(z - x_n)^2 - 4y_n}}{2}. \tag{3.3}$$

It is easy to check that

$$\sqrt{-\det\left(J_n - y_n J_n^{-1}\right)} = \sqrt{(z - x_n)^2 - 4y_n} = \alpha_n - y_n\alpha_n^{-1}. \tag{3.4}$$

Lemma 3.2. *For J_n defined by (2.11), there are ($k = 1, 2, \ldots$)*

$$J_n^k - y_n^k J_n^{-k} = \frac{\alpha_n^k - y_n^k\alpha_n^{-k}}{\alpha_n - y_n\alpha_n^{-1}}\left(J_n - y_n J_n^{-1}\right), \tag{3.5}$$

where

$$\frac{\alpha_n^k - y_n^k\alpha_n^{-k}}{\alpha_n - y_n\alpha_n^{-1}} = \frac{1}{2^{k-1}}\sum_{s=0}^{[\frac{k-1}{2}]}\binom{k}{2s+1}(z - x_n)^{k-2s-1}((z - x_n)^2 - 4y_n)^s. \tag{3.6}$$

Proof. By (3.2) and (3.3), there is

$$J_n + y_n J_n^{-1} = \left(\alpha_n + y_n\alpha_n^{-1}\right)I, \tag{3.7}$$

which implies that $J_n^2 - y_n^2 J_n^{-2} = \left(\alpha_n + y_n\alpha_n^{-1}\right)\left(J_n - y_n J_n^{-1}\right)$. Then (3.5) is true for $k = 1$ and 2.

Suppose (3.5) is true for $k - 1$ and k. Let us show that it is true for $k + 1$. Multiplying (3.5) with (3.7), we have

$$J_n^{k+1} - y_n^{k+1} J_n^{-k-1} + y_n\left(J_n^{k-1} - y_n^{k-1} J_n^{-k+1}\right)$$
$$= \frac{\alpha_n^{k+1} - y_n^{k+1}\alpha_n^{-k-1}}{\alpha_n - y_n\alpha_n^{-1}}\left(J_n - y_n J_n^{-1}\right) + y_n\frac{\alpha_n^{k-1} - y_n^{k-1}\alpha_n^{-k+1}}{\alpha_n - y_n\alpha_n^{-1}}\left(J_n - y_n J_n^{-1}\right).$$

By the assumption, equation (3.5) is true for $k + 1$.

By (3.3) and $y_n\alpha_n^{-1} = \frac{1}{2}(z - x_n - ((z - x_n)^2 - 4y_n)^{1/2})$, there is

$$\alpha_n^k - y_n^k\alpha_n^{-k} = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(z - x_n)^{k-j}\left(((z - x_n)^2 - 4y_n)^{\frac{j}{2}} - (-1)^j((z - x_n)^2 - 4y_n)^{\frac{j}{2}}\right)$$
$$= \frac{1}{2^{k-1}}\sum_{s=0}^{[\frac{k-1}{2}]}\binom{k}{2s+1}(z - x_n)^{k-2s-1}((z - x_n)^2 - 4y_n)^{s+\frac{1}{2}},$$

where the terms with even j are canceled, and the terms with odd j are combined by taking $j = 2s + 1$. □

Let

$$f_{2m-2}(z) = \frac{1}{2}\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1}\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r}\frac{y_n^r}{2^{q-2r-1}} f^{(q,r)}(z), \tag{3.8}$$

where

$$f^{(q,r)}(z) = \sum_{s=0}^{[\frac{q-2r-1}{2}]}\binom{q-2r}{2s+1}(z - x_n)^{q-2r-2s-1}((z - x_n)^2 - 4y_n)^s. \tag{3.9}$$

The discussions above imply the following result.

Proposition 3.1. *If x_n, y_n, and t_j $(j = 1, \ldots, 2m)$ satisfy equation (2.27), then for any $z \in \mathbb{C}$ (complex plane), there is*

$$D_n^{-1}\tilde{A}_n(z)D_n = f_{2m-2}(z)\big(J_n(z) - y_n J_n^{-1}(z)\big), \tag{3.10}$$

where $\tilde{A}_n(z)$ is defined by (2.12), $f_{2m-2}(z)$ is a polynomial of degree $2m-2$ defined by (3.8) and (3.9), and $J_n(z)$ is defined by (2.11).

4. Asymptotics as $z \to \infty$

Proposition 4.1. *If x_n, $y_n(>0)$, and t_j $(j = 1, \ldots, 2m)$ satisfy equations (2.26) and (2.27), then for $z \in \mathbb{C}\setminus[x_n - 2\sqrt{y_n}, x_n + 2\sqrt{y_n}]$, there is*

$$\sqrt{-\det \tilde{A}_n} = \frac{1}{2}\sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1}\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r} y_n^r\big(\alpha_n^{q-2r} - \big(y_n\alpha_n^{-1}\big)^{q-2r}\big). \tag{4.1}$$

As $z \to \infty$ in the complex plane, there is the asymptotics

$$\sqrt{-\det \tilde{A}_n(z)} = \frac{1}{2}V'(z) - \frac{n}{z} + O\left(\frac{1}{z^2}\right), \tag{4.2}$$

where $V(z) = \sum_{j=0}^{2m} t_j z^j$, $t_{2m} > 0$ and $' = \partial/\partial z$.

Proof. As $z \to \infty$, there is $D_n^{-1}\tilde{A}_n(z)D_n \sim m t_{2m} z^{2m-1}\,\mathrm{diag}(-1, 1)$ by (2.10), (2.11) and (2.12). Since $t_{2m} > 0$, the branch of the square root is determined by $(-\det \tilde{A}_n(z))^{1/2} \sim m t_{2m} z^{2m-1}$, as $z \to +\infty$ on the real line. Then (3.5) with $k = q - 2r$ and (3.4) imply

$$\sqrt{-\det\big(J_n^{q-2r} - \big(y_n J_n^{-1}\big)^{q-2r}\big)} = \alpha_n^{q-2r} - \big(y_n\alpha_n^{-1}\big)^{q-2r},$$

which gives (4.1) according to (3.1). Here we denote $\sum_{r=0}^{-1}\cdot = 0$ when $q = 0$ for convenience in the discussions.

Let $s = q - r = ([q/2] - \mu_q - r) + [q/2] + 1$ in the terms $\big(y_n\alpha_n^{-1}\big)^{q-2r}$ in (4.1). Then

$$\sqrt{-\det \tilde{A}_n} = \frac{1}{2}\sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1}\left[\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{r}\alpha_n^{q-r}\big(y_n\alpha_n^{-1}\big)^r - \sum_{s=[q/2]+1}^{q}\binom{q}{s}\alpha_n^{q-s}\big(y_n\alpha_n^{-1}\big)^s\right].$$

By the binomial formula, there is

$$\sqrt{-\det \tilde{A}_n} = \frac{1}{2}\sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1}\binom{j-1}{q} x_n^{j-q-1}\left[\big(\alpha_n + y_n\alpha_n^{-1}\big)^q - \mu_q\binom{q}{[q/2]}\alpha_n^{q-[q/2]}\big(y_n\alpha_n^{-1}\big)^{[q/2]} - 2\sum_{s=[q/2]+1}^{q}\binom{q}{s} y_n^s\alpha_n^{-(2s-q)}\right].$$

Since $\alpha_n + y_n\alpha_n^{-1} = z - x_n$, the first part in the bracket above gives $\frac{1}{2}V'(z)$ by considering the outside summations. The second part in the bracket can be dropped off by using (2.27). For

$s = [q/2] + 1$ in the third part in the bracket, we have the following by separating the odd q and even q terms, and by noting that q starts from $q = 1$, and j starts from $j = 2$ for this part,

$$\sum_{j=1}^{2m} jt_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \binom{q}{[q/2]+1} y_n^{[q/2]+1} \alpha_n^{q-2[q/2]-2}$$

$$= \alpha_n^{-1} \sum_{j=2}^{2m} jt_j \sum_{p=0}^{[\frac{j}{2}]-1} \binom{j-1}{2p+1} \binom{2p+1}{p} x_n^{j-2p-2} y_n^{p+1}$$

$$+ \alpha_n^{-2} \sum_{j=2}^{2m} jt_j \sum_{p=1}^{[\frac{j-1}{2}]} \binom{j-1}{2p} \binom{2p}{p+1} x_n^{j-2p-1} y_n^{p+1},$$

where $q = 2p + 1$ when q is odd, and $q = 2p$ when q is even. As $z \to \infty$, it is easy to check that $\alpha_n^{-1} = z^{-1} + O(z^{-2})$. Combining the discussions above, we get

$$\sqrt{-\det(\tilde{A}_n)} = \frac{1}{2} \sum_{j=1}^{2m} jt_j z^{j-1} - \frac{n}{z} + O\left(\frac{1}{z^2}\right),$$

by using (2.26). □

In the following, we show that $(-\det A_n(z))^{1/2}$ has similar asymptotics as discussed for $(-\det \tilde{A}_n(z))^{1/2}$ as $z \to \infty$. Since the restriction conditions for A_n and $\tilde{A}_n$ are different in the asymptotics, separate proofs are needed. The proof in the following adopts the Cauchy kernel used in [10, 26].

Proposition 4.2. *For A_n defined by (2.8) with $n \geqslant 2m$, as $z \to \infty$, there is*

$$\sqrt{-\det A_n(z)} = \frac{1}{2} V'(z) - \frac{n}{z} + O\left(\frac{1}{z^2}\right), \tag{4.3}$$

when the parameters satisfy (2.22).

Proof. Denote

$$\hat{p}_n(z) = \int_{-\infty}^{\infty} \frac{e^{-V(z')}}{z'-z} p_n(z')\, dz' \qquad \text{and} \qquad \Psi_n = \begin{pmatrix} p_n & \hat{p}_n \\ p_{n-1} & \hat{p}_{n-1} \end{pmatrix}.$$

It is not hard to see that $V'(z)$ and $F_n(z)$ are both of degree $2m - 1$ in z. Since $n \geqslant 2m$, by orthogonality there is

$$\int_{-\infty}^{\infty} \frac{e^{-V(z')}}{z'-z} \left[D_n(F(z') - F_n(z))D_n^{-1} - (V'(z') - V'(z))\right] \begin{pmatrix} p_n(z') \\ p_{n-1}(z') \end{pmatrix} dz' = 0.$$

Then it can be verified that

$$\frac{\partial}{\partial z}\Psi_n = D_n F_n D_n^{-1} \Psi_n - \Psi_n \operatorname{diag}(0, V').$$

Multiplying Ψ_n^{-1} on both sides of the above equation and taking trace, we get the following by using $\partial \det \Psi_n / \partial z = 0$,

$$\operatorname{tr} F_n(z) = V'(z), \tag{4.4}$$

which implies $-\det A_n(z) = \frac{1}{4}(V'(z))^2 - \det F_n(z)$.

According to (2.7), $D_n F_n$ can be expressed as

$$\left[C_{n-1} \bar{J}_{n-1}^{-1} \cdots \bar{J}_{n-m+1}^{-1} + \cdots + C_{n-2m-1} \bar{J}_{n-2m+2} \cdots \bar{J}_{n-m}\right] \bar{J}_{n-m+1} \cdots \bar{J}_{n-1} \bar{J}_n.$$

Considering the leading terms as $z \to \infty$, we have

$$D_n F_n = [\det(\bar{J}_{n-1} \cdots \bar{J}_{n-m+1})^{-1} z^{m-1} \operatorname{diag}(a_{n,n-1} h_{n-1}, 0) + \cdots \\ + z^{m-1} \operatorname{diag}(0, a_{n-1,n-2m} h_{n-2m})] \bar{J}_{n-m+1} \cdots \bar{J}_{n-1} \bar{J}_n.$$

It can be calculated by (2.6) that $a_{n-1,n-2m} h_{n-2m} = 2mt_{2m} h_{n-1}$. Since $\det D_n = h_n h_{n-1}$, and $v_n = h_n / h_{n-1}$, there is $\det F_n = 2mt_{2m} a_{n,n-1} z^{2m-2}(1 + O(z^{-1}))$. By (2.22), there is

$$\det F_n(z) = 2mnt_{2m} z^{2m-2}(1 + O(z^{-1})). \tag{4.5}$$

Then (4.3) is proved. □

5. Density and related problems

For the density on one interval, denote $z/n^{\frac{1}{2m}}, t_j/n^{1-\frac{j}{2m}}, x_n/n^{\frac{1}{2m}}$, and $y_n/n^{\frac{1}{m}}$ by η, g_j, a, and b^2 respectively according to the universality argument [10], where $b > 0$. Let $\alpha_n = n^{\frac{1}{2m}}\alpha$, and then $y_n\alpha_n^{-1} = n^{\frac{1}{2m}}(b^2\alpha^{-1})$, where $\alpha = (\eta - a + \sqrt{(\eta - a)^2 - 4b^2})/2$, and $b^2\alpha^{-1} = (\eta - a - \sqrt{(\eta - a)^2 - 4b^2})/2$. By proposition 3.1, it follows that for $z \in \mathbb{C}\diagdown[x_n - 2\sqrt{y_n}, x_n + 2\sqrt{y_n}]$,

$$\sqrt{-\det \tilde{A}_n(z)} = n^{1-\frac{1}{2m}} k_{2m-2}(\eta)\sqrt{(\eta - a)^2 - 4b^2}, \qquad \eta \in \mathbb{C}\diagdown[a - 2b, a + 2b],$$

where

$$k_{2m-2}(\eta) = \sum_{j=1}^{2m} jg_j \sum_{q=0}^{j-1} \binom{j-1}{q} a^{j-q-1} \sum_{r=0}^{[\frac{q}{2}]-\mu_q} \binom{q}{r} \frac{b^{2r}}{2^{q-2r}} k^{(q,r)}(\eta), \tag{5.1}$$

and

$$k^{(q,r)}(\eta) = \sum_{s=0}^{[\frac{q-2r-1}{2}]} \binom{q-2r}{2s+1} (\eta - a)^{q-2r-2s-1}((\eta - a)^2 - 4b^2)^s. \tag{5.2}$$

Define an analytic function

$$\omega_m(\eta) = k_{2m-2}(\eta)\sqrt{(\eta - a)^2 - 4b^2}, \qquad \eta \in \mathbb{C}\diagdown[a - 2b, a + 2b]. \tag{5.3}$$

The parameters a, b, and g_j ($j = 1, \ldots, 2m$) are restricted to satisfy the following conditions:

$$\sum_{j=2}^{2m} jg_j \sum_{p=0}^{[\frac{j}{2}]-1} \binom{j-1}{2p+1} \binom{2p+1}{p} a^{j-2p-2} b^{2p+2} = 1, \tag{5.4}$$

$$\sum_{j=1}^{2m} jg_j \sum_{p=0}^{[\frac{j-1}{2}]} \binom{j-1}{2p} \binom{2p}{p} a^{j-2p-1} b^{2p} = 0. \tag{5.5}$$

These two conditions (5.4) and (5.5) are obtained from (2.26) and (2.27). By proposition 4.1, if a, b, and g_j ($j = 1, \ldots, 2m$) satisfy equations (5.4) and (5.5), then for $\eta \in \mathbb{C}\diagdown[a-2b, a+2b]$ there is

$$\omega_m(\eta) = \frac{1}{2} \sum_{j=1}^{2m} jg_j \sum_{q=0}^{j-1} \binom{j-1}{q} a^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} b^{2r}(\alpha^{q-2r} - (b^2\alpha^{-1})^{q-2r}). \tag{5.6}$$

As $\eta \to \infty$,

$$\omega_m(\eta) = \frac{1}{2}W'(\eta) - \frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right). \tag{5.7}$$

In (5.6), the index j actually starts from $j = 2$, and index q starts from 1. We keep this form just for convenience in the later discussion for free energy when we use equation (5.5) where j is from $j = 1$ and p is from $p = 0$. Let

$$\rho_m(\eta) = \frac{1}{\pi} k_{2m-2}(\eta)\sqrt{(\eta_+ - \eta)(\eta - \eta_-)}, \qquad \eta \in [\eta_-, \eta_+], \tag{5.8}$$

where $\eta_- = a - 2b$, $\eta_+ = a + 2b$, $b > 0$ and $k_{2m-2}(\eta)$ is given by (5.1). By (5.3) and (5.8), there is

$$\omega_m(\eta)|_{[\eta_-,\eta_+]^\pm} = \pm\pi\mathrm{i}\rho_m(\eta)|_{[\eta_-,\eta_+]}, \tag{5.9}$$

where $[\eta_-, \eta_+]^+$ and $[\eta_-, \eta_+]^-$ stand for the upper and lower edges of the interval $[\eta_-, \eta_+]$ respectively. Since $\rho_m(\eta)$ is non-negative, we also need

$$k_{2m-2}(\eta) \geqslant 0, \tag{5.10}$$

for $\eta \in [\eta_-, \eta_+]$.

For the density on multiple disjoint intervals, consider

$$J^{(\nu)} = \begin{pmatrix} 0 & 1 \\ -b_1^2 & \eta - a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -b_\nu^2 & \eta - a_\nu \end{pmatrix}, \tag{5.11}$$

where $\nu \geqslant 1$. According to the Cayley–Hamilton theorem for $J^{(\nu)}$, choose $\alpha^{(\nu)} = (\Lambda + \sqrt{\Lambda^2 - 4b^{(\nu)2}})/2$, where $\Lambda = \Lambda(\eta) = \operatorname{tr} J^{(\nu)}$, $b^{(\nu)} > 0$ and $b^{(\nu)^2} = \det J^{(\nu)}$. We can transform $g_j\ (j = 1, \ldots, 2m)$ into a new set of parameters $g'_j\ (j = 1, \ldots, 2m)$ by a linear transformation so that $W'(\eta) = \sum_{s=0}^{\nu-1} \eta^s \sum_{q=0}^{m_s} g'_{\nu q+s+1} \Lambda^q$ for some integers $m_s\ (s = 0, \ldots, \nu - 1)$ as done in section 2.2.

Define another analytic function

$$\omega_m^{(\nu)}(\eta) = \frac{1}{2}\sum_{s=0}^{\nu-1} \eta^s \sum_{q=1}^{m_s} g'_{\nu q+s+1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} b^{(\nu)^{2r}} \left(\alpha^{(\nu)^{q-2r}} - \left(b^{(\nu)^2}\alpha^{(\nu)^{-1}}\right)^{q-2r}\right), \tag{5.12}$$

for η in outside of the cuts to be discussed in the following. Then there is $\omega_m^{(\nu)}(\eta) = \frac{1}{2}W'(\eta) - X(\eta)$, where

$$X(\eta) = \sum_{s=0}^{\nu-1} \eta^s \sum_{q=0}^{m_s} g'_{\nu q+s+1} \left[\frac{\mu_q}{2}\binom{q}{[q/2]} b^{(\nu)^{2[q/2]}} \alpha^{(\nu)^{q-2[q/2]}} + \sum_{r=[q/2]+1}^{q} \binom{q}{r} b^{(\nu)^{2r}} \alpha^{(\nu)^{q-2r}}\right]. \tag{5.13}$$

It is the same argument as discussed for $\omega_m(\eta)$ that if the parameters satisfy the conditions

$$\sum_{p=0}^{[\frac{m_{\nu-1}-1}{2}]} g'_{2\nu p+2\nu} \binom{2p+1}{p} b^{(\nu)^{2p+2}} = 1, \tag{5.14}$$

$$\sum_{p=0}^{[\frac{m_s}{2}]} g'_{2\nu p+s+1} \binom{2p}{p} b^{(\nu)^{2p}} = 0, \tag{5.15}$$

for $s = 0, 1, \ldots, \nu - 1$, then

$$\omega_m^{(\nu)}(\eta) = \frac{1}{2}W'(\eta) - \frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right), \tag{5.16}$$

as $\eta \to \infty$.

Now, consider the cuts for $\omega_m^{(\nu)}(\eta)$, determined by $\alpha^{(\nu)} - b^{(\nu)2}\alpha^{(\nu)-1} = \sqrt{\Lambda^2 - 4b^{(\nu)2}}$. The equation $\Lambda^2 - 4b^{(\nu)2} = 0$ has 2ν roots, real or complex. If there is a complex root, its complex conjugate is also a root. If there is repeated root, the factor can be moved out from the inside of the square root in the expression of $\omega_m^{(\nu)}(\eta)$. Therefore, without loss of generality, we consider that the equation $\Lambda^2 - 4b^{(\nu)2} = 0$ has $2\nu_1$ simple real roots $\eta_-^{(s)}, \eta_+^{(s)}$, $s = 1, \ldots, \nu_1$, and $2\nu_2$ simple complex roots $\eta_s, \bar{\eta}_s$, $s = 1, \ldots, \nu_2$, where $\bar{\eta}_s$ is the complex conjugate of η_s, $\operatorname{Im}\eta_s > 0$, and $\nu = \nu_1 + \nu_2$. Suppose the real roots are so ordered that $[\eta_-^{(s)}, \eta_+^{(s)}]$, $s = 1, \ldots, \nu_1$, form a set of disjoint intervals, $\Omega = \cup_{s=1}^{\nu_1}[\eta_-^{(s)}, \eta_+^{(s)}]$. Define

$$\rho_m^{(\nu)}(\eta) = \frac{1}{\pi}\operatorname{Re}\frac{1}{i}\omega_m^{(\nu)}(\eta)|_{\Omega^+}, \tag{5.17}$$

for $\eta \in \Omega$. It can be seen that when $\nu = \nu_1 = 1$, $\omega_m^{(1)} = \omega_m$, $\rho_m^{(1)}(\eta) = \rho_m(\eta)$, and conditions (5.14) and (5.15) become (5.4) and (5.5) respectively.

Choose ν_2 points $\eta_s^{(0)}$ on the real line outside of Ω, such that the straight lines Γ_s's, each one connecting η_s and $\eta_s^{(0)}$ for $s = 1, \ldots, \nu_2$, do not intersect each other. Now, $\omega_m^{(\nu)}(\eta)$ is well defined and analytic in the outside of $\Omega \cup \cup_{s=1}^{\nu_2}(\Gamma_s \cup \bar{\Gamma}_s)$, where $\bar{\Gamma}_s$ is the straight line connecting $\bar{\eta}_s$ and $\eta_s^{(0)}$. Let Γ_s^* be the closed counterclockwise contour along the edges of $\Gamma_s \cup \bar{\Gamma}_s$, and define

$$I_s = \int_{\Gamma_s^*} \omega_m^{(\nu)}(\eta)\,\mathrm{d}\eta, \qquad \text{and} \qquad \hat{I}_s(\eta) = \int_{\Gamma_s^*} \frac{\omega_m^{(\nu)}(\lambda)}{\lambda - \eta}\,\mathrm{d}\lambda, \qquad \eta \in \Omega,$$

for $s = 1, \ldots, \nu_2$. According to the definition of Γ_s^*, I_s and $\hat{I}_s(\eta)$ are real.

Proposition 5.1. *If the parameters a_s, b_s ($s = 1, \ldots, \nu$), and g_j ($j = 1, \ldots, 2m$) satisfy conditions (5.14) and (5.15), then $\rho_m^{(\nu)}(\eta)$ defined by (5.17) on Ω satisfies (1.3) and (1.4).*

Proof. Let Γ be a large counterclockwise circle of radius R, and Ω^* be the union of closed counterclockwise contours around the upper and lower edges of all the intervals in Ω. Then by the Cauchy theorem and (5.16),

$$\int_{\Omega^*}\left(\omega_m^{(\nu)}(\eta) - \frac{1}{2}W'(\eta)\right)\mathrm{d}\eta + \sum_{s=1}^{\nu_2} I_s = \int_{\Gamma}\left(\omega_m^{(\nu)}(\eta) - \frac{1}{2}W'(\eta)\right)\mathrm{d}\eta \to -2\pi\mathrm{i},$$

as $R \to \infty$, which implies $\int_\Omega \rho_m(\eta)\,\mathrm{d}\eta = 1$ by (5.9), $\int_{\Omega^*} W'(\eta)\,\mathrm{d}\eta = 0$, and I_s are real. So $\rho_m(\eta)$ satisfies condition (1.3).

Change the Ω^- and Ω^+ discussed above just at $\eta \in \Omega$ as semicircles of ϵ radius. By (5.16) and $\int_{\Gamma_s^*}\frac{W'(\lambda)}{\lambda-\eta}\,\mathrm{d}\lambda = 0$, there is

$$\frac{1}{2\pi\mathrm{i}}\int_{\Omega^*}\frac{\omega_m^{(\nu)}(\lambda) - \frac{1}{2}W'(\lambda)}{\lambda - \eta}\,\mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\sum_{s=1}^{\nu_2}\hat{I}_s = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\omega_m^{(\nu)}(\lambda) - \frac{1}{2}W'(\lambda)}{\lambda - \eta}\,\mathrm{d}\lambda \to 0,$$

as $R \to \infty$. Then taking the real part on both sides and by (5.17), we get

$$\frac{1}{2}W'(\eta) = \frac{1}{2\pi}\int_{\Omega^*}\frac{\operatorname{Re}\frac{1}{\mathrm{i}}\omega_m^{(\nu)}(\lambda)}{\lambda - \eta}\,\mathrm{d}\lambda \to (\mathrm{P})\int_\Omega \frac{\rho_m(\lambda)}{\eta - \lambda}\,\mathrm{d}\lambda,$$

as $\epsilon \to 0$. □

By the discussions above, it can be seen that when $\nu_2 = 0$, a_s, b_s ($s = 1, \ldots, \nu$), and g_j ($j = 1, \ldots, 2m$) satisfy relations (5.14) and (5.15), $G(\eta) = \omega_m^{(\nu)}(\eta) - \frac{1}{2}W'(\eta)$ solves the scalar Riemann–Hilbert problem [10]:

$$\begin{aligned} &\text{(i)} \quad G(\eta) \text{ is analytic when } \eta \in \mathbb{C}\setminus\Omega; \\ &\text{(ii)} \quad G(\eta)|_{\Omega^+} + G(\eta)|_{\Omega^-} = -W'(\eta); \\ &\text{(iii)} \quad G(\eta) \to 0, \qquad \text{as} \quad \eta \to \infty. \end{aligned} \tag{5.18}$$

In other words, if a_s and b_s can be chosen such that

$$(\operatorname{tr} J^{(\nu)})^2 - 4\det J^{(\nu)} = \prod_{j=1}^{\nu} \left(\eta - \eta_-^{(j)}\right)\left(\eta - \eta_+^{(j)}\right), \tag{5.19}$$

then the corresponding Riemann–Hilbert problem can be well solved, where the left-hand side of (5.19) is also equal to $-\det\left(J^{(\nu)} - \left(\det J^{(\nu)}\right) J^{(\nu)^{-1}}\right)$.

Meanwhile, by proposition 4.2, when $n \geqslant 2m$ and the parameters satisfy (2.22), $\sigma_n(z)$ defined by

$$\sigma_n(z) = \frac{1}{\pi} \operatorname{Re}\sqrt{\det A_n(z)}, \qquad -\infty < z < \infty, \tag{5.20}$$

satisfies $\int_{-\infty}^{\infty} \sigma_m(z)\,\mathrm{d}z = n$ and $(\mathrm{P})\int_{-\infty}^{\infty} \frac{\sigma_m(z')}{z-z'}\,\mathrm{d}z' = \frac{1}{2}V'(z)$, which is the level density [5]. When the density involves the parameter n, the discrete Painlevé I equation and the initial conditions when n is less than $2m$ need to be considered to calculate the functions u_n and v_n.

6. Free energy for the one interval case

Lemma 6.1. *For $\rho_m(\eta)$ defined by (5.8) on $[\eta_-, \eta_+]$ with the parameters a, b and g_j $(j = 1, \ldots, 2m)$ satisfying conditions (5.10), (5.4) and (5.5), there is*

$$\int_{\eta_-}^{\eta_+} \eta^k \rho_m(\eta)\,\mathrm{d}\eta = \sum_{j=2}^{2m} j g_j \sum_{q=1}^{j-1} \binom{j-1}{q} a^{j-q-1} b^{q+1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{[q/2]+r+1} R_{2r+\mu_q+1,k}, \tag{6.1}$$

where

$$R_{l,k} = \frac{\mathrm{i}}{\pi} \int_{-\pi}^{\pi} (a + 2b\cos\theta)^k\, \mathrm{e}^{-\mathrm{i}l\theta} \sin\theta\,\mathrm{d}\theta, \tag{6.2}$$

with $l = 2r + \mu_q + 1$ and $\mu_q = (1 + (-1)^q)/2$.

Proof. Let Ω^* be the closed counterclockwise contour around lower and upper edges of $[\eta_-, \eta_+]$, and Γ be a large counterclockwise circle. Since Ω^* is counterclockwise, by (5.9) and the Cauchy theorem we have

$$\int_{\eta_-}^{\eta_+} \eta^k \rho_m(\eta)\,\mathrm{d}\eta = -\frac{1}{2\pi\mathrm{i}} \int_{\Omega^*} \eta^k \omega_m(\eta)\,\mathrm{d}\eta = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \eta^k \omega_m(\eta)\,\mathrm{d}\eta.$$

So the problem becomes the calculation of the integral $\int_\Gamma \eta^k \omega_m(\eta)\,\mathrm{d}\eta$.

By using binomial formula skill as in the proof of proposition 4.1, and $\int_\Gamma \eta^k(\alpha + b^2\alpha^{-1})^q\,\mathrm{d}\eta = \int_\Gamma \eta^k(\eta - a)^q\,\mathrm{d}\eta = 0$, we can obtain

$$\int_{\eta_-}^{\eta_+} \eta^k \rho_m(\eta)\,\mathrm{d}\eta = \frac{1}{2\pi\mathrm{i}} \sum_{j=2}^{2m} j g_j \sum_{q=1}^{j-1} \binom{j-1}{q} a^{j-q-1} \sum_{s=[q/2]+1}^{q} \binom{q}{s} b^{2s} \int_\Gamma \eta^k \alpha^{-(2s-q)}\,\mathrm{d}\eta. \tag{6.3}$$

Note that the index q is changed to start from 1, and j is changed to start from 2.

On Ω^*, there is $\eta = a + 2b\cos\theta, -\pi \leqslant \theta \leqslant \pi$, where $a = (\eta_+ + \eta_-)/2$ and $2b = (\eta_+ - \eta_-)/2 > 0$. Then $\alpha^{-1} = b^{-1}\,\mathrm{e}^{-\mathrm{i}\theta}$, where the square root takes positive and negative imaginary value on upper and lower edge of $[\eta_-, \eta_+]$ respectively. By the Cauchy theorem, the integral along Γ can be changed to along Ω^*, which implies

$$\int_\Gamma \eta^k \alpha^{-(2s-q)}\,\mathrm{d}\eta = -2b^{q-2s+1}\int_{-\pi}^{\pi}(a+2b\cos\theta)^k\,\mathrm{e}^{-\mathrm{i}(2s-q)\theta}\sin\theta\,\mathrm{d}\theta.$$

Let $r = s - [q/2] - 1$ in (6.3). Because the range of s is from $[q/2]+1$ to q, and $q = 2[q/2] - \mu_q + 1$, the range of r is from 0 to $[q/2] - \mu_q$. Since $2s - q = 2r + \mu_q + 1$, this lemma is proved. □

Lemma 6.2. *For $\rho_m(\eta)$ defined by (5.8) on $[\eta_-, \eta_+]$ with the parameters a, b and g_j $(j = 1, \ldots, 2m)$ satisfying conditions (5.10), (5.4) and (5.5), there is*

$$\int_{\eta_-}^{\eta_+}\ln|\eta-a|\rho_m(\eta)\,\mathrm{d}\eta$$
$$= \ln(2b) - \sum_{j=2}^{2m} jg_j \sum_{q=1}^{j-1}\binom{j-1}{q}a^{j-q-1}b^{q+1}\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{[q/2]+r+1}\Theta_{2r+\mu_q+1}, \tag{6.4}$$

where

$$\Theta_l = \mathrm{Re}\frac{\mathrm{i}}{\pi}\int_0^{\pi}\theta\,\mathrm{e}^{\mathrm{i}\theta}[(\mathrm{e}^{\mathrm{i}\theta}+\sqrt{\mathrm{e}^{2\mathrm{i}\theta}-1})^l - (\mathrm{e}^{\mathrm{i}\theta}-\sqrt{\mathrm{e}^{2\mathrm{i}\theta}-1})^l]\,\mathrm{d}\theta, \tag{6.5}$$

with $l = 2r + \mu_q + 1$ and $\mu_q = (1 + (-1)^q)/2$.

Proof. Let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be a closed counterclockwise contour, where γ_1 is the upper edges of $[\eta_-, a]$, γ_2 is the upper edges of $[a, \eta_+]$ and γ_3 is the semi-circle of radius $2b$ with center a. Applying the Cauchy theorem for $\ln(\eta - a)\omega_m(\eta)$, we have

$$\int_{\gamma_1}(\ln|\eta-a|+\pi\mathrm{i})\omega_m(\eta)\,\mathrm{d}\eta + \int_{\gamma_2}\ln|\eta-a|\omega_m(\eta)\,\mathrm{d}\eta + \int_{\gamma_3}\ln(2b\,\mathrm{e}^{\mathrm{i}\theta})\omega_m(\eta)\,\mathrm{d}\eta = 0.$$

When $\eta \in \gamma_1 \cup \gamma_2$, $\omega(\eta) = \pi\mathrm{i}\rho_m(\eta)$. Then taking the imaginary part for the above equation, we get

$$\int_{\eta_-}^{\eta_+}\ln|\eta-a|\rho_m(\eta)\,\mathrm{d}\eta - \ln(2b) + \frac{1}{\pi}\mathrm{Re}\int_{\gamma_3}\theta\omega_m(\eta)\,\mathrm{d}\eta = 0, \tag{6.6}$$

where we have used $\int_{\gamma_3}\omega_m(\eta)\,\mathrm{d}\eta = -\int_{\gamma_1\cup\gamma_2}\omega_m(\eta)\,\mathrm{d}\eta = -\pi\mathrm{i}\int_{\gamma_1\cup\gamma_2}\rho_m(\eta)\,\mathrm{d}\eta = -\pi\mathrm{i}$. So the problem becomes the calculation of the integral $\int_{\gamma_3}\theta\omega_m(\eta)\,\mathrm{d}\eta$.

Rewrite (5.6) as

$$\omega_m(\eta) = \frac{1}{2}\sum_{j=2}^{2m} jg_j\sum_{q=1}^{j-1}\binom{j-1}{q}a^{j-q-1}\sum_{s=0}^{[q/2]-\mu_q}\binom{q}{s}b^{2s}(\alpha^{q-2s} - (b^2\alpha^{-1})^{q-2s}).$$

Let $r = [q/2] - \mu_q - s$. The range of r is from 0 to $[q/2] - \mu_q$. Since $q = 2[q/2] - \mu_q + 1$, and $q - 2s = 2([q/2] - \mu_q - s) + \mu_q + 1$, we have the following:

$$\omega_m(\eta) = \frac{1}{2}\sum_{j=2}^{2m} jg_j\sum_{q=1}^{j-1}\binom{j-1}{q}a^{j-q-1}$$
$$\times\sum_{r=0}^{[q/2]-\mu_q}\binom{q}{[q/2]+r+1}b^{2([q/2]-\mu_q-r)}(\alpha^{2r+\mu_q+1} - (b^2\alpha^{-1})^{2r+\mu_q+1}).$$

On γ_3, we have $\eta - a = 2b\,\mathrm{e}^{\mathrm{i}\theta}$, which implies $\alpha = b\big(\mathrm{e}^{\mathrm{i}\theta} + \sqrt{\mathrm{e}^{2\mathrm{i}\theta} - 1}\big)$, and $b^2\alpha^{-1} = b\big(\mathrm{e}^{\mathrm{i}\theta} - \sqrt{\mathrm{e}^{2\mathrm{i}\theta} - 1}\big)$. It follows that

$$\int_{\gamma_3} \theta(\alpha^{2r+\mu_q+1} - (b^2\alpha^{-1})^{2r+\mu_q+1})\,\mathrm{d}\eta$$

$$= 2\mathrm{i}b^{2r+\mu_q+2} \int_0^{\pi} \theta\,\mathrm{e}^{\mathrm{i}\theta}\big[(\mathrm{e}^{\mathrm{i}\theta} + \sqrt{\mathrm{e}^{2\mathrm{i}\theta} - 1})^{2r+\mu_q+1} - (\mathrm{e}^{\mathrm{i}\theta} - \sqrt{\mathrm{e}^{2\mathrm{i}\theta} - 1})^{2r+\mu_q+1}\big]\,\mathrm{d}\theta.$$

We finally have

$$\frac{1}{\pi}\,\mathrm{Re}\int_{\gamma_3} \theta\omega_m(\eta)\,\mathrm{d}\eta = \sum_{j=2}^{2m} jg_j \sum_{q=1}^{j-1} \binom{j-1}{q} a^{j-q-1} b^{q+1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{[q/2]+r+1} \Theta_{2r+\mu_q+1}.$$

Then by (6.6), the lemma is proved. □

The Θ_l in the above lemma can be solved by some elementary integrals using the recursion method as described in the following.

Lemma 6.3. *For $k = 0, 1, 2, \ldots$, there are*

$$\int_0^{\pi} \theta\,\mathrm{e}^{2\mathrm{i}\theta}(1 - \mathrm{e}^{2\mathrm{i}\theta})^{k+\frac{1}{2}}\,\mathrm{d}\theta = \frac{\pi}{(2k+3)\mathrm{i}}, \tag{6.7}$$

$$\int_0^{\pi} \theta\,\mathrm{e}^{\mathrm{i}\theta}(1 - \mathrm{e}^{2\mathrm{i}\theta})^{k+\frac{1}{2}}\,\mathrm{d}\theta = -2\int_0^1\int_0^1 (1 - x^2y^2)^{k+\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}y + \frac{\pi\mathrm{i}}{2} B\left(\frac{1}{2}, k + \frac{3}{2}\right), \tag{6.8}$$

where $B(\cdot,\cdot)$ is the Euler beta function.

Proof. The first equation in this lemma can be easily verified by using integration by parts,

$$\int_0^{\pi} \theta\,\mathrm{e}^{2\mathrm{i}\theta}(1 - \mathrm{e}^{2\mathrm{i}\theta})^{k+\frac{1}{2}}\,\mathrm{d}\theta = \frac{1}{(2k+3)\mathrm{i}}\int_0^{\pi} (1 - \mathrm{e}^{2\mathrm{i}\theta})^{k+\frac{3}{2}}\,\mathrm{d}\theta = \frac{\pi}{(2k+3)\mathrm{i}}.$$

To prove the second equation, consider

$$J(\gamma) = \int_0^{\pi} \mathrm{e}^{\mathrm{i}\theta}(1 - \gamma\,\mathrm{e}^{2\mathrm{i}\theta})^{k+\frac{1}{2}}\,\mathrm{d}\theta, \qquad \text{and} \qquad I(\gamma) = \int_0^{\pi} \theta\,\mathrm{e}^{\mathrm{i}\theta}(1 - \gamma\,\mathrm{e}^{2\mathrm{i}\theta})^{k+\frac{1}{2}}\,\mathrm{d}\theta,$$

for $0 \leqslant \gamma \leqslant 1$. It can be calculated that $\big(\gamma^{\frac{1}{2}}J(\gamma)\big)' = \mathrm{i}\gamma^{-\frac{1}{2}}(1-\gamma)^{k+\frac{1}{2}}$, where $' = \mathrm{d}/\mathrm{d}\gamma$. Then $\gamma^{\frac{1}{2}}J(\gamma) = \mathrm{i}\int_0^{\gamma} t^{-\frac{1}{2}}(1-t)^{k+\frac{1}{2}}\,\mathrm{d}t$, which implies

$$J(\gamma) = 2\mathrm{i}\int_0^1 (1 - \gamma x^2)^{k+\frac{1}{2}}\,\mathrm{d}x, \tag{6.9}$$

by taking $t = \gamma x^2$.

It can be calculated that $\big(\gamma^{\frac{1}{2}}I(\gamma)\big)' = \frac{\pi\mathrm{i}}{2}\gamma^{-\frac{1}{2}}(1-\gamma)^{k+\frac{1}{2}} - \frac{1}{2\mathrm{i}}\gamma^{-\frac{1}{2}}J(\gamma)$. Then by (6.9) and taking integral from 0 to 1, we have

$$I(1) = \frac{\pi\mathrm{i}}{2}\int_0^1 \gamma^{-\frac{1}{2}}(1-\gamma)^{k+\frac{1}{2}}\,\mathrm{d}\gamma - \int_0^1 \gamma^{-\frac{1}{2}}\left(\int_0^1 (1 - \gamma x^2)^{k+\frac{1}{2}}\,\mathrm{d}x\right)\mathrm{d}\gamma,$$

which gives the second equation in this lemma by taking $\gamma = y^2$ in the last term above. □

To further calculate the real part of the right-hand side of equation (6.8), consider the following line and double integrals for $k = 0, 1, 2, \ldots$,

$$l_k = \int_0^1 (1 - x^2)^{k+\frac{1}{2}}\,\mathrm{d}x, \qquad \text{and} \qquad \mathrm{d}_k = \int_0^1\int_0^1 (1 - x^2y^2)^{k+\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}y.$$

First, $l_0 = \frac{\pi}{4}$, and

$$d_0 = \frac{1}{2}\int_0^1 \left(\sqrt{1-y^2} + \frac{1}{y}\sin^{-1} y\right) \mathrm{d}y = \frac{\pi}{8} + \frac{\pi}{4}\ln 2. \tag{6.10}$$

When $k \geqslant 1$, by integration by parts, we can verify that l_k satisfy a recursion formula $l_k = l_{k-1} - \frac{1}{2k+1} l_k$, which gives $l_k = \frac{(2k+1)!!}{(2k+2)!!}\frac{\pi}{2}$. Also by integration by parts, there is $d_k = d_{k-1} + \frac{1}{2k+1}(l_k - d_k)$, which implies

$$d_k = \frac{2k+1}{2k+2} d_{k-1} + \frac{(2k+1)!!}{(2k+2)!!}\frac{\pi}{4(k+1)}. \tag{6.11}$$

Specially

$$d_1 = \frac{9\pi}{64} + \frac{3\pi}{16}\ln 2, \tag{6.12}$$

which will be used in the non-symmetric density discussed in section 7. By combining the above results, we have the following result for the free energy.

Proposition 6.1. *For the eigenvalue density $\rho_m(\eta)$ defined by (5.8) with the parameters a, b, and g_j ($j = 1, \dots, 2m$) satisfying conditions (5.10), (5.4) and (5.5), there is the following formula for the free energy (1.1):*

$$E^{(0)} = \frac{1}{2} W(a) - \ln(2b) + \sum_{j=2}^{2m} j g_j \sum_{q=1}^{j-1} \binom{j-1}{q} a^{j-1-q} b^{q+1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{[q/2]+r+1} E_{2r+\mu_q+1}, \tag{6.13}$$

where

$$E_l = \frac{1}{2}\sum_{k=0}^{2m} g_k R_{l,k} + \Theta_l, \tag{6.14}$$

with $l = 2r + \mu_q + 1$ and $\mu_q = (1 + (-1)^q)/2$.

Proof. Consider equation (1.4). By taking integral from a to η for the variable η, we have $\int_{\eta_-}^{\eta_+} \ln|\lambda - \eta| \rho_m(\lambda)\,\mathrm{d}\lambda = \frac{1}{2}W(\eta) - \frac{1}{2}W(a) + \int_{\eta_-}^{\eta_+} \ln|\lambda - a|\rho_m(\lambda)\,\mathrm{d}\lambda$. Multiplying $\rho_m(\eta)$ and taking $\int_{\eta_-}^{\eta_+} \mathrm{d}\eta$ on both sides of this equation, we get by using (1.3)

$$\int_{\eta_-}^{\eta_+}\int_{\eta_-}^{\eta_+} \ln|\lambda - \eta|\rho_m(\lambda)\rho_m(\eta)\,\mathrm{d}\lambda\,\mathrm{d}\eta = \frac{1}{2}\int_{\eta_-}^{\eta_+} W(\eta)\rho_m(\eta)\,\mathrm{d}\eta - \frac{1}{2}W(a) + \int_{\eta_-}^{\eta_+} \ln|\lambda - a|\rho_m(\lambda)\,\mathrm{d}\lambda.$$

Then according to (1.1), we arrive at

$$E^{(0)} = \frac{1}{2}W(a) + \frac{1}{2}\sum_{k=0}^{2m} g_k \int_{\eta_-}^{\eta_+} \eta^k \rho_m(\eta)\,\mathrm{d}\eta - \int_{\eta_-}^{\eta_+} \ln|\eta - a|\rho_m(\eta)\,\mathrm{d}\eta.$$

By lemmas 6.1 and 6.2, the integrals above can be expressed in terms of $R_{l,k}$ and Θ_l. After simplifications, the result is proved. □

7. Some special densities

7.1. The model for $m = 2$

When $m = 2$, or $W(\eta) = g_0 + g_1\eta + g_2\eta^2 + g_3\eta^3 + g_4\eta^4$, proposition 5.1 gives the general eigenvalue density on one interval

$$\rho_2(\eta) = \frac{1}{2\pi}(2g_2 + 3g_3(\eta + a) + 4g_4(\eta^2 + a\eta + a^2 + 2b^2))\sqrt{4b^2 - (\eta - a)^2}, \tag{7.1}$$

where the parameters satisfy the following conditions:

$$2g_2 + 3g_3(\eta + a) + 4g_4(\eta^2 + a\eta + a^2 + 2b^2) \geqslant 0, \qquad \eta \in [\eta_-, \eta_+], \tag{7.2}$$

$$2g_2b^2 + 6g_3ab^2 + 12g_4(a^2 + b^2)b^2 = 1, \tag{7.3}$$

$$g_1 + 2g_2a + 3g_3(a^2 + 2b^2) + 4g_4a(a^2 + 6b^2) = 0. \tag{7.4}$$

The free energy function is given by proposition 6.1:

$$E^{(0)} = W(a) + \tfrac{3}{4} - \ln b - 4g_4b^4 - 6(g_3 + 4g_4a)^2b^6 - 6g_4^2b^8. \tag{7.5}$$

When $g_1 = g_2 = 0$, i.e. $W(\eta) = g_0 + g_3\eta^3 + g_4\eta^4$, the conditions become

$$3g_3(\eta + a) + 4g_4(\eta^2 + a\eta + a^2 + 2b^2) \geqslant 0, \qquad \eta \in [\eta_-, \eta_+], \tag{7.6}$$

$$g_3 = -\frac{8a(a^2 + 6b^2)}{3b^2(5a^4 + 3(a^2 - 4b^2)^2)}, \tag{7.7}$$

$$g_4 = \frac{2(a^2 + 2b^2)}{b^2(5a^4 + 3(a^2 - 4b^2)^2)}. \tag{7.8}$$

Condition (7.6) is satisfied if and only if $\tau = \frac{4b^2}{a^2}$ is restricted in the intervals $0 < \tau \leqslant \tau_-$ or $\tau_+ \leqslant \tau$, where $\tau_+ = 1 + \sqrt{5}$, and τ_- is uniquely determined by the conditions $0 < \tau_- < 1/2$ and $1 - 2\tau_-^{1/2} + \frac{3}{4}\tau_-^2 = 0$. Approximately we have $\tau_- \approx 0.28$ and $\tau_+ \approx 3.24$. The corresponding free energy function is reduced to

$$E^{(0)} = g_0 + \frac{3}{8} - \ln b - \frac{8}{3\tau\bar{\tau}} - \frac{15\tau + 32}{3\bar{\tau}} - \frac{140\tau - 40}{3\bar{\tau}^2}, \tag{7.9}$$

for $\tau \in (0, \tau_-] \cup [\tau_+, \infty)$, where $\bar{\tau} = 5 + 3(1 - \tau)^2$. The density function in this case can be further changed into the following forms. Let $\eta = ax$ and $\tau = c^2 (c > 0)$. Then

$$\rho_2(\eta)\,\mathrm{d}\eta = \frac{16}{\pi}\frac{\left(\frac{x}{c} - \frac{c}{2}\right)^2 + \frac{x^2-1}{2}}{5 + 3(1 - c^2)^2}\sqrt{c^2 - (x - 1)^2}\,\mathrm{d}x, \tag{7.10}$$

for $x \in [1 - c, 1 + c]$, where $c \in (0, c_-] \cup [c_+, \infty)$, $c_- = \sqrt{\tau_-}$ and $c_+ = \sqrt{\tau_+}$. On the other hand, if $\eta = -ax$ and $\tau = c^2 (c > 0)$, then

$$\rho_2(\eta)\,\mathrm{d}\eta = \frac{16}{\pi}\frac{\left(\frac{x}{c} + \frac{c}{2}\right)^2 + \frac{x^2-1}{2}}{5 + 3(1 - c^2)^2}\sqrt{c^2 - (x + 1)^2}\,\mathrm{d}x, \tag{7.11}$$

for $x \in [-1 - c, -1 + c]$, where $c \in (0, c_-] \cup [c_+, \infty)$, $c_- = \sqrt{\tau_-}$ and $c_+ = \sqrt{\tau_+}$.

The density on two disjoint intervals can be calculated by using the method discussed before. Briefly, there is

$$\rho_2^{(2)}(\eta) = \frac{1}{2\pi}(3g_3 + 4g_4(a_1 + a_2 + \eta))\mathrm{Re}\sqrt{4b_1^2b_2^2 - \left((\eta - a_1)(\eta - a_2) - b_1^2 - b_2^2\right)^2}, \tag{7.12}$$

where $-\infty < \eta < \infty$, and

$$4g_4b_1^2b_2^2 = 1, \tag{7.13}$$

$$2g_2 + (3g_3 + 4g_4(a_1 + a_2))(a_1 + a_2) - 4g_4\left(a_1a_2 - b_1^2 - b_2^2\right) = 0, \tag{7.14}$$

$$g_1 - (3g_3 + 4g_4(a_1 + a_2))\left(a_1a_2 - b_1^2 - b_2^2\right) = 0. \tag{7.15}$$

It can be checked that if we take $a_1 = a_2 = a$ and $b_1 = b_2 = b$ in the above, then a and b satisfy equations (7.3) and (7.4). In addition, the parameters need to satisfy extra condition(s) such that $\rho_2^{(2)}(\eta)$ does not take a negative value. Relevant discussions can be found in [15] if ones are interested in the corresponding free energy.

As a remark, if $W(\eta) = g_3\eta^3 + g_4\eta^4$ is degenerated to $W(\eta) = g_4\eta^4$ by taking $a \to 0$, (7.9) becomes $E^{(0)} = 3/8 - \ln b$. We will see next that $E^{(0)}$ has the same result as $W(\eta) = g_2\eta^2 + g_4\eta^4$ is degenerated to $W(\eta) = g_4\eta^4$. We can also use (7.1) to get other special densities for $m = 2$. The density formula (7.1) and conditions (7.3) and (7.4) for $g_1 = g_4 = 0$, $g_2 = 1/2$ coincide with results (45) and (46) in [6].

7.2. Symmetric densities

For symmetric densities, consider $W(\eta) = g_0 + g_2\eta^2 + \cdots + g_{2m}\eta^{2m}$, and $a = 0$.

When $m = 2$, there is

$$\rho_2(\eta) = \frac{1}{\pi}(g_2 + 2g_4(\eta^2 + 2b^2))\sqrt{4b^2 - \eta^2}, \tag{7.16}$$

for $\eta \in [-2b, 2b]$, with the restriction conditions

$$g_2 + 2g_4(\eta^2 + 2b^2) \geqslant 0, \qquad \eta \in [-2b, 2b], \tag{7.17}$$

$$2g_2b^2 + 12g_4b^4 = 1. \tag{7.18}$$

The free energy becomes

$$E^{(0)} = g_0 + \frac{3}{4} - \ln b + \frac{1}{24}(2g_2b^2 - 1)(9 - 2g_2b^2), \tag{7.19}$$

which agrees with equation (1.5) obtained in [6] if we choose $g_2 = 1/2$. If $E^{(0)}$ is taken as a function of $2g_2b^2$, it has a singular point at $2g_2b^2 = 2$, or at $g_4 = g_4^c$, where $g_4^c = -\frac{g_2^2}{12}$. This singular point is corresponding to the bound for condition (7.17) with $g_4 < 0$, as well as the singularity $\nu/(\nu - 1)$ with $\nu = 2$ in [14] (theorem 2.3). In fact, for the first part, $g_2 + 2g_4(4b^2 + 2b^2) = 0$ and (7.18) imply $12g_4 = -g_2^2$. For the second part, if $g_2 = 1/2$ in (7.18), then $-12v^2\frac{\mathrm{d}g_4}{\mathrm{d}v} = 2\left(\frac{1}{v} - \frac{1}{2}\right)$, which implies $v = 2$ if $\mathrm{d}g_4/\mathrm{d}v = 0$, where $v = b^2$.

When $m = 3$, by proposition 5.1, there is

$$\rho_3(\eta) = \frac{1}{\pi}(g_2 + 2g_4(\eta^2 + 2b^2) + 3g_6(\eta^4 + 2b^2\eta^2 + 6b^4))\sqrt{4b^2 - \eta^2},$$

for $\eta \in [-2b, 2b]$, and (7.2) and (7.3) become

$$g_2 + 2g_4(\eta^2 + 2b^2) + 3g_6(\eta^4 + 2b^2\eta^2 + 6b^4) \geqslant 0, \qquad \eta \in [-2b, 2b],$$

$$2g_2b^2 + 12g_4b^4 + 60g_6b^6 = 1.$$

Generally, the density is

$$\rho_m(\eta) = \frac{1}{\pi}k_{2m-2}(\eta)\sqrt{4b^2 - \eta^2}, \qquad \eta \in [-2b, 2b], \tag{7.20}$$

where

$$k_{2m-2}(\eta) = \sum_{j=1}^{m} j g_{2j} \sum_{p=1}^{j} \binom{2j-1}{j-p} \frac{b^{2(j-p)}}{4^{p-1}} \sum_{s=0}^{p-1} \binom{2p-1}{2s+1} \eta^{2(p-s-1)} (\eta^2 - 4b^2)^s, \tag{7.21}$$

and

$$k_{2m-2}(\eta) \geqslant 0, \qquad \eta \in [-2b, 2b], \tag{7.22}$$

$$\sum_{j=1}^{m} 2j g_{2j} \binom{2j-1}{j} b^{2j} = 1. \tag{7.23}$$

Here, formula (7.21) is obtained from (5.1) and (5.2) by choosing $g_1 = g_3 = \cdots = g_{2m-1} = 0$, $a = 0$, and then replacing j by $2j$, and taking $q = 2j - 1$ and $r = j - p$.

By (5.7), there is for large $R > 0$

$$k_{2m-2}(\eta) = \frac{1}{2\pi \mathrm{i}} \oint_{|\lambda|=R} \frac{\omega_m(\lambda)}{\sqrt{\lambda^2 - 4b^2}} \frac{\mathrm{d}\lambda}{\lambda - \eta} = \frac{1}{2\pi \mathrm{i}} \oint_{|\lambda|=R} \frac{\frac{1}{2}W'(\lambda)}{\sqrt{\lambda^2 - 4b^2}} \frac{\mathrm{d}\lambda}{\lambda - \eta}.$$

When $W(\eta) = g_{2m}\eta^{2m}$, there is $k_{2m-2}(\eta) = m g_{2m} h(\eta)$, where $h(\eta)$ is given by (1.9) which is from (6.151) in [10], and (7.23) becomes (1.10).

Appendix. Densities in other models

A.1. Density associated with Laguerre polynomials

Consider the Laguerre polynomials $L_n^{(\alpha)}(x)$ [19],

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha \mathrm{e}^{-x}\, \mathrm{d}x = \Gamma(\alpha + 1) \binom{n+\alpha}{n} \delta_{m,n},$$

where $\alpha > -1$, and $\Gamma(\cdot)$ is the Gamma function. Choose

$$\Phi_n(x) = \mathrm{e}^{-x/2} x^{\alpha/2} \big(L_n^{(\alpha)}(x),\ L_{n-1}^{(\alpha)}(x)\big)^T.$$

It can be verified that $\Phi_n(x)$ satisfies the following equation [19, 27]:

$$\frac{\partial}{\partial x}\Phi_n = A_n(x)\Phi_n,$$

where

$$A_n(x) = \frac{1}{x} \begin{pmatrix} -\frac{x-\alpha}{2} + n & -n-\alpha \\ n & \frac{x-\alpha}{2} - n \end{pmatrix},$$

and $\operatorname{tr} A_n(x) = 0$. It can be calculated that

$$\sqrt{\det(A_n)} = \frac{n}{2x} \sqrt{\left(\left(1 + \sqrt{\frac{n+\alpha}{n}}\right)^2 - \frac{x}{n}\right) \left(\frac{x}{n} - \left(1 - \sqrt{\frac{n+\alpha}{n}}\right)^2\right)}. \tag{A.1}$$

Let $x = n\lambda$, $q = \frac{n}{n+\alpha}$, $\lambda_+ = \big(1 + \frac{1}{\sqrt{q}}\big)^2$ and $\lambda_- = \big(1 - \frac{1}{\sqrt{q}}\big)^2$. Then

$$\frac{1}{(n+\alpha)\pi} \sqrt{\det(A_n(x))}\, \mathrm{d}x = \frac{q}{2\pi\lambda} \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}\, \mathrm{d}\lambda, \tag{A.2}$$

which gives the density obtained in [22, 23], and the density is used in econophysics and relevant researches for studying the distribution of the positive eigenvalues, for example, see [35, 36].

A.2. Densities associated with the polynomials on a unit circle

Consider the orthogonal polynomials $p_n(z) = z^n + \cdots$ on the unit circle $|z| = 1$ with the potential function $V(z) = s(z + z^{-1})$ [4, 37]:

$$\oint p_m(z)\overline{p_n(z)}\,\mathrm{e}^{s(z+1/z)}\frac{\mathrm{d}z}{2\pi\mathrm{i}z} = h_n\delta_{m,n},$$

where the integral is on $|z| = 1$, and $\overline{p_n(z)}$ is the complex conjugate of $p_n(z)$. On the unit circle, let

$$\Phi_n(z) = \mathrm{e}^{\frac{s}{2}(z+1/z)}\left(z^{-n/2}p_n(z), z^{n/2}\overline{p_n(z)}\right)^T.$$

Then by equation (4.10) in [4], there is

$$\frac{\partial}{\partial z}\Phi_n = A_n(z)\Phi_n,$$

where

$$A_n(z) = \begin{pmatrix} \dfrac{s}{2} + \dfrac{s}{2z^2} + \dfrac{n - 2sx_nx_{n+1}}{2z} & s\left(x_{n+1} - \dfrac{x_n}{z}\right)z^{-1} \\ s\left(x_n - \dfrac{x_{n+1}}{z}\right) & -\dfrac{s}{2} - \dfrac{s}{2z^2} - \dfrac{n - 2sx_nx_{n+1}}{2z} \end{pmatrix},$$

$\operatorname{tr} A_n(z) = 0$, and $x_n (= p_n(0))$ satisfies the discrete Painlevé II equation

$$\frac{n}{s}x_n = -\left(1 - x_n^2\right)(x_{n+1} + x_{n-1}), \tag{A.3}$$

with $x_n \in [-1, 1]$. Then

$$\sqrt{\det(A_n)} = \frac{1}{\mathrm{i}}\sqrt{\left(\frac{s}{2} + \frac{s}{2z^2} + \frac{n - 2sx_nx_{n+1}}{2z}\right)^2 + \frac{s^2}{z}\left(x_n - \frac{x_{n+1}}{z}\right)\left(x_{n+1} - \frac{x_n}{z}\right)}. \tag{A.4}$$

Let $n/s = \lambda$ and $u_n = -x_{n+1}/x_n$. Then $\lambda\big/\left(1 - x_n^2\right) = u_n + 1/u_{n-1}$, or asymptotically as $n, s \to \infty$,

$$u_n \sim \left[\frac{\lambda}{2\left(1 - x_n^2\right)} + \sqrt{\left(\frac{\lambda}{2\left(1 - x_n^2\right)}\right)^2 - 1}\,\right]^{-1}.$$

If $\lambda = 2\left(1 - x_n^2\right)(\leqslant 2)$, then $u_n \sim 1$, or $x_{n+1} \sim -x_n \sim x_{n-1}$,

$$\frac{1}{n\pi}\sqrt{\det(A_n(z))}\,\mathrm{d}z \sim \frac{2}{\pi\lambda}\cos\frac{\alpha}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{\alpha}{2}}\,\mathrm{d}\alpha, \tag{A.5}$$

where $z = \mathrm{e}^{\mathrm{i}\alpha}$, which gives the density (29) in [20] for weak coupling, and if $\lambda > 2$, then $u_n < 1$, or $x_n \to 0$,

$$\frac{1}{n\pi}\sqrt{\det(A_n(z))}\,\mathrm{d}z \sim \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos\alpha\right)\mathrm{d}\alpha, \tag{A.6}$$

which gives the density (24) in [20] for strong coupling. It was obtained in [20] that the free energy for this model has continuous first- and second-order derivatives with respect to λ, and the third-order derivative is discontinuous at the critical point $\lambda = 2$ or $n/s = 2$. At this critical point, the discrete Painlevé II equation can be reduced to the Painlevé II equation as discussed in the Riemann–Hilbert problem [38].

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